

The model companion of the class of pseudocomplemented semilattices is finitely axiomatizable

Joël Adler*

Institute of Mathematics, University of Bern, Switzerland

February 27, 2013

Abstract

We show that the class \mathcal{PCSL}^{ec} of existentially closed pseudocomplemented semilattices is finitely axiomatizable by appropriately extending the finite axiomatization of the class \mathcal{PCSL}^{ac} of algebraically closed pseudocomplemented semilattices presented in [8]. Because \mathcal{PCSL}^{ec} coincides with the model companion of the class \mathcal{PCSL} of pseudocomplemented semilattices this addendum to [8] solves the problem posed by Albert and Burris in the final paragraph of [1]: “Does the class of pseudocomplemented semilattices have a finitely axiomatizable model companion?”

1 Introduction

The notion of existential closedness is motivated by the notion of an algebraically closed field. In the class of fields existential and algebraic closedness coincide: If \mathbf{K} is a field and $p(\vec{x})$ and $q(\vec{x})$ are polynomials over \mathbf{K} , then the satisfiability of the negated equation $p(\vec{x}) \neq q(\vec{x})$ is equivalent to the satisfiability of the equation $x \cdot (p(\vec{x}) - q(\vec{x})) = 1$. Thus every system of negated equations over \mathbf{K} can be replaced by a system of equations.

However, the following examples show that this is not the general situation: In the class of boolean algebras every boolean algebra is algebraically closed whereas a boolean algebra \mathbf{B} is existentially closed if and only if \mathbf{B} is atomfree. An abelian group \mathbf{G} is algebraically closed if and only if \mathbf{G} is divisible, whereas \mathbf{G} is existentially closed if and only if \mathbf{G} is divisible and contains an infinite direct sum of copies of \mathbb{Q}/\mathbb{Z} (as a module). For a more detailed description of the notion of algebraic and existential closedness we refer the reader to [6].

As \mathcal{PCSL} is a finitely generated universal Horn class with both the amalgamation and joint embedding property \mathcal{PCSL} has a model companion, see [1] for details. The model companion need not exist with the class of groups serving as an example. Furthermore, we have that if the set Σ of $\mathcal{L}_{\mathcal{PCSL}}$ -sentences is the model companion of \mathcal{PCSL} , then the class of models of Σ is exactly \mathcal{PCSL}^{ec} . Thus, proving that \mathcal{PCSL}^{ec} is finitely

*joel.adler@phbern.ch

axiomatizable solves the problem posed by Albert and Burris in the final paragraph of [1].

An axiomatizable class of \mathcal{L} -structures is *finitely axiomatizable* if and only if both the class itself as well as its complementary class are closed under elementary equivalence and ultraproducts. Instead of proving that \mathcal{PCSL}^{ec} and its complementary class are both closed under elementary equivalence and ultraproducts we specify a finite list of $\mathcal{L}_{\mathcal{PCSL}}$ -sentences that axiomatize \mathcal{PCSL}^{ec} .

2 Basic properties of pseudocomplemented semilattices and notation

A *pseudocomplemented semilattice* $\langle P; \wedge, *, 0 \rangle$ is an algebra where $\langle P; \wedge \rangle$ is a meet-semilattice with least element 0, and for all $x, y \in P$, $x \wedge a = 0$ if and only if $x \leq a^*$. Obviously, $1 := 0^*$ is the greatest element of P . We define $x \parallel y$ to hold if neither $x \leq y$ nor $y \leq x$ holds. An element d of P satisfying $d^* = 0$ is called *dense*, and if additionally $d \neq 1$ holds, then d is called a *proper dense* element. For $\mathbf{P} \in \mathcal{PCSL}$ the set $D(\mathbf{P})$ denotes the subset of dense elements of \mathbf{P} , $\langle D(\mathbf{P}); \wedge \rangle$ being a filter of $\langle P; \wedge \rangle$. An element s is called *skeletal* if $s^{**} = s$. The subset of skeletal elements of \mathbf{P} is denoted by $\text{Sk}(\mathbf{P})$. The abuse of notation $\text{Sk}(x)$ for $x \in \text{Sk}(\mathbf{P})$ should not cause ambiguities. Obviously, $\text{Sk}(\mathbf{P}) = \{x^* : x \in P\}$. In $\text{Sk}(\mathbf{P})$ the supremum of two elements exists with $\sup_{\text{Sk}}\{a, b\} = (a^* \wedge b^*)^*$ for $a, b \in \text{Sk}(\mathbf{P})$. Instead of $\sup_{\text{Sk}}\{a, b\}$ we use the shorter $a \dot{\vee} b$, assuming $a, b \in \text{Sk}(\mathbf{P})$. Observe that $\langle \text{Sk}(\mathbf{P}); \wedge, \dot{\vee}, *, 0, 1 \rangle$ is a boolean algebra. In the subset $\text{Sk}(\mathbf{P})$ of skeletal elements we consider the subset $C(\mathbf{P}) := \{c \in \text{Sk}(\mathbf{P}) : x \geq c \ \& \ x \geq c^* \rightarrow x = 1\}$ of *central* elements of \mathbf{P} . Finally, the set of all atoms of a pseudocomplemented semilattice \mathbf{P} is denoted by $\text{At}(\mathbf{P})$.

For any pseudocomplemented semilattice \mathbf{P} the pseudocomplemented semilattice $\hat{\mathbf{P}}$ is obtained from \mathbf{P} by adding a new top element. The maximal dense element of $\hat{\mathbf{P}}$ different from 1 is denoted by e . Furthermore, the PCSLs $\hat{\mathbf{B}}$ with \mathbf{B} being a boolean algebra are exactly the subdirectly irreducible PCSLs. Moreover, let $\mathbf{2}$ denote the two-element boolean algebra, $\mathbf{3}$ the three-element p-algebra $\{0, e, 1\}$ and \mathbf{A} the countable atomfree boolean algebra. For a survey of pseudocomplemented semilattices consult [2] or [5].

For a p-semilattice \mathbf{P} and an arbitrary element $a \in P$ the binary relation $x \sim_a y : \iff a \wedge x = a \wedge y$ is a congruence. The factor algebra \mathbf{P} / \sim_a , in the sequel denoted by the shorter $(\mathbf{P})_a$, is isomorphic to $\langle \{a \wedge x : x \in P\}; \cdot, ', 0, a \rangle$, where $(a \wedge x) \cdot (a \wedge y)$ is defined by $a \wedge (x \wedge y)$ and $(a \wedge x)'$ by $a \wedge x^*$. Given the direct product $\prod_{i=1}^n \mathbf{P}_i$ and $a = (0, \dots, 0, 1, \dots, 1)$ with the first k places being 0, the factor algebra $(\prod_{i=1}^n \mathbf{P}_i)_a$ is isomorphic to $\prod_{i=k+1}^n \mathbf{P}_i$. Furthermore, the map $f_a : P \rightarrow (P)_a$ defined by $f_a(x) = a \wedge x$ is a surjective homomorphism.

We use $\mathbf{Q} \leq \mathbf{P}$ (respectively, $\mathbf{P} \geq \mathbf{Q}$) freely to indicate that \mathbf{Q} is a subalgebra of \mathbf{P} in whatever signature \mathbf{P} and \mathbf{Q} are being considered.

Finally, we need the notion of a homomorphism over a set: Let \mathbf{P} and \mathbf{Q} be p-semilattices, $\{a_1, \dots, a_m\}$ a subset of $P \cap Q$. We say a homomorphism $f : \mathbf{P} \rightarrow \mathbf{Q}$ is *over* $\{a_1, \dots, a_m\}$ if $f(a_i) = a_i$ holds for all $1 \leq i \leq m$.

For more background on p-semilattices in general consult [2] and [5], for the notions concerning the problem tackled in this paper consult [8].

3 The class \mathcal{PCSL}^{ac}

On various occasions we will use the following — semantic — characterization of algebraically closed p-semilattices, established in [11].

Theorem 3.1. *A p-semilattice \mathbf{P} is algebraically closed if and only if for any finite subalgebra $\mathbf{S} \leq \mathbf{P}$ there exist $r, s \in \mathbb{N}$ and a p-semilattice \mathbf{S}' isomorphic to $2^r \times (\hat{\mathbf{A}})^s$ such that $\mathbf{S} \leq \mathbf{S}' \leq \mathbf{P}$.*

In [8] the following list of axioms is introduced to axiomatize the class of algebraically closed p-semilattices.

Definition 3.2. Let \mathbf{P} be a p-semilattice. \mathbf{P} will be said to satisfy

(AC1) if

$$(\forall a, b, c)(\exists x, y)(c \geq a \wedge b \rightarrow (x \geq a \ \& \ y \geq b \ \& \ x \wedge y = c)),$$

(AC2) if

$$(\forall a, b, c, t)(\exists x)((a^* = b^* = c^* = 0 \ \& \ c < b < a \ \& \ t \wedge c < t \wedge b < t \wedge a) \rightarrow (c < x < a \ \& \ x \wedge b = c \ \& \ t \wedge c < t \wedge x < t \wedge a')),$$

(AC3) if

$$\begin{aligned} &(\forall d, d_m, f, f_m, x, k)(\exists z_k)((d \in D(\mathbf{P}) \ \& \ d_m \in D(\mathbf{P}) \ \& \ k \in \text{Sk}(\mathbf{P}) \ \& \\ & \quad d \parallel d_m \ \& \ f \leq d_m \ \& \ f_m \leq d \ \& \ f_m \not\leq d_m \ \& \ k \leq d \ \& \\ & \quad k^* \wedge f \not\leq d \ \& \ x^* \leq d_m) \rightarrow \\ & \quad (z_x \in \text{Sk}(\mathbf{P}) \ \& \ k \leq z_x \leq d \ \& \ z_x^* \wedge f \not\leq d \ \& \\ & \quad z_x \wedge f_m \not\leq d_m \ \& \ z_x \wedge x^* \leq d_m)), \end{aligned}$$

(AC4) if

$$\begin{aligned} &(\forall d, b_1)(\exists b_2)((d \in D(\mathbf{P}) \ \& \ b_2 \in \text{Sk}(\mathbf{P}) \ \& \ b_1 < d < 1) \rightarrow \\ & \quad (b_2 \in \text{Sk}(\mathbf{P}) \ \& \ b_1 < b_2 < d \ \& \ b_1 \dot{\vee} b_2^* < d)). \end{aligned}$$

The following theorem, the main result of [8], states that the preceding list of axioms together with a finite axiomatization of the class \mathcal{PCSL} is a finite axiomatization of the class \mathcal{PCSL}^{ac} .

Theorem 3.3. *A p-semilattice \mathbf{P} is algebraically closed if and only if \mathbf{P} satisfies axioms (AC1)–(AC4).*

4 A finite axiomatization of \mathcal{PCSL}^{ec}

Theorem 4.7 states that the list of axioms (EC1)–(EC5) below together with the axioms (AC1)–(AC4), which axiomatize \mathcal{PCSL}^{ac} , axiomatize \mathcal{PCSL}^{ec} . Its proof consists of carrying out the following steps:

- We will first show that a p-semilattice \mathbf{P} is existentially closed if and only if there is for every finite subalgebra \mathbf{S} extendable to a finite subalgebra \mathbf{T} within an extension \mathbf{Q} of \mathbf{P} a subalgebra \mathbf{S}' of \mathbf{P} isomorphic to \mathbf{T} over \mathbf{S} .

- Apply Theorem 3.1 to obtain that \mathbf{S} and \mathbf{T} may be assumed to be direct products of subdirectly irreducible p-semilattices.
- Apply Lemma 4.3 to obtain that \mathbf{S} may be assumed to be a single subdirectly irreducible p-semilattice.
- Apply Lemmata 4.4 and 4.5 to determine what a chain $(\mathbf{T}_i)_{1 \leq i \leq n}$ of subalgebras \mathbf{T}_i of \mathbf{Q} such that $\mathbf{T}_1 = \mathbf{S}$, $\mathbf{T}_n = \mathbf{T}$ and $\mathbf{T}_i \leq \mathbf{T}_{i+1}$, $i = 1, \dots, n-1$, looks like.
- The application of Lemma 4.6 yields that if there is such a chain in \mathbf{Q} there is a chain $(\mathbf{S}_i)_{1 \leq i \leq n}$ in \mathbf{P} such that \mathbf{S}_i and \mathbf{T}_i are isomorphic over S for $1 \leq i \leq n$.

Definition 4.1. Let \mathbf{P} be a p-semilattice. \mathbf{P} will be said to satisfy

(EC1) if

$$(\forall b_1, b_2)(\exists b_3)((\text{Sk}(b_1) \ \& \ \text{Sk}(b_2) \ \& \ b_1 < b_2) \rightarrow (\text{Sk}(b_3) \ \& \ b_1 < b_3 < b_2)),$$

(EC2) if

$$(\forall b_1, d)(\exists b_2)((\text{Sk}(b_1) \ \& \ \text{D}(d) \ \& \ b_1 < d \ \& \ b_1^* \parallel d) \rightarrow (\text{Sk}(b_2) \ \& \ b_1 < b_2 \parallel d \ \& \ b_2 < 1 \ \& \ b_1 \dot{\vee} b_2^* < d \ \& \ b_1^* \wedge b_2 \parallel d)),$$

(EC3) if

$$(\exists d)(\text{D}(d) \ \& \ d < 1),$$

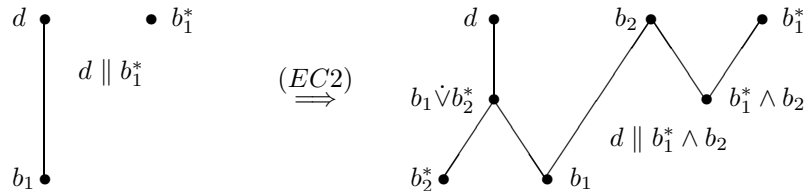
(EC4) if

$$(\forall d_1, d_2)(\exists d_3)((\text{D}(d_1) \ \& \ d_1 < d_2) \rightarrow (d_1 < d_3 < d_2)),$$

(EC5) if

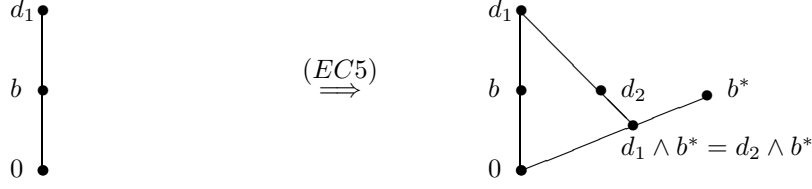
$$(\forall b, d_1)(\exists d_2)((\text{D}(d_1) \ \& \ \text{Sk}(b) \ \& \ 0 < b < d_1) \rightarrow (\text{D}(d_2) \ \& \ d_2 < d_1 \ \& \ b \parallel d_2 \ \& \ d_1 \wedge b^* = d_2 \wedge b^*)).$$

A couple of sentences to explain what the axioms (EC1)–(EC5) mean are appropriate. (EC1) and (EC4) are the usual density conditions holding in existentially closed posets. Skeletal and dense elements must be mentioned separately because $b_1 < b_3 < b_2$ with b_1 and b_2 skeletal does not imply that b_3 is skeletal as well. (EC3) simply guarantees the existence of a nontrivial dense element. Clearly an existentially closed p-algebra must contain a nontrivial dense element since any p-algebra can be embedded into a p-algebra with a nontrivial dense element. To understand (EC2) and (EC5) diagrams may be helpful.



(EC2) ensures that a finite subalgebra $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, $1 \leq f(i)$, of a p-semilattice \mathbf{P} satisfying (EC2) can be extended in \mathbf{P} to a subalgebra \mathbf{S}' isomorphic to \mathbf{T} over S for an arbitrary subalgebra $\mathbf{T} \cong \mathbf{2} \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ of an

extension \mathbf{Q} of \mathbf{P} . Applying (EC2) to suitable $d, b_1 \in S$ yields a skeletal element b_2 that behaves with respect to \mathbf{S} as the element $(0, 1, \dots, 1) \in T \setminus S$.



(EC5) ensures that a finite subalgebra $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, $1 \leq f(i)$, of a p-semilattice \mathbf{P} satisfying (EC5) can be extended in \mathbf{P} to a subalgebra \mathbf{S}' isomorphic to \mathbf{T} over S for an arbitrary subalgebra $\mathbf{T} \cong \prod_{i=1}^{q+1} \widehat{\mathbf{F}_{f(i)}}$, $f(q+1) > 0$ and $\min D(T) < \min D(S)$, of an extension \mathbf{Q} of \mathbf{P} . Applying (EC5) to suitable $d_1, b \in S$ yields a dense element d_2 that behaves with respect to \mathbf{S} as the element $(e, \dots, e) \in T \setminus S$.

Remark 4.2. 1. Observe in (EC3) that $d^* = 0$ & $d < e$ implies $e^* = 0$ as $D(\mathbf{P})$ is a filter of \mathbf{P} .

2. Let \mathbf{P} be a p-algebra satisfying (EC1). Then the subalgebra $\text{Sk}(\mathbf{P})$ is atomfree and thus existentially closed in $\text{Sk}(\mathbf{Q})$ for any p-algebra \mathbf{Q} extending \mathbf{P} .

Lemma 4.3. Let \mathbf{P}_i , $i \in I$, be p-semilattices and $\mathbf{P} = \prod_{i \in I} \mathbf{P}_i$. Then any of the axioms (AC1)–(AC4) and (EC1)–(EC5) holds in \mathbf{P} if and only if it holds in every \mathbf{P}_i ($i \in I$).

Proof. Straightforward. \square

To prove the central theorem of this paper we need three more lemmata. The first two lemmata are semantic statements about how a finite direct product of finite subdirectly irreducible p-semilattices contains a subdirectly irreducible p-semilattice respectively a product of subdirectly irreducible p-semilattices as a subalgebra. The third lemma is syntactic in the sense that it states how in a p-semilattices satisfying (AC1)–(AC4) and (EC1)–(EC5) a finite subdirectly irreducible subalgebra with a non-trivial dense element can be extended to a finite direct product of finite subdirectly irreducible p-semilattices.

Lemma 4.4. If $\mathbf{T} = \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, $q \geq 1$, $f(i) \geq 1$, and $\mathbf{S} \leq \mathbf{T}$ with $\mathbf{S} \cong \widehat{\mathbf{F}_s}$, $s \geq 0$, then there is a sequence of subalgebras $\mathbf{T}_0, \dots, \mathbf{T}_{2q}$ of \mathbf{T} satisfying

- $\mathbf{T}_0 = \mathbf{S}$,
- $\mathbf{T}_k \leq \mathbf{T}_{k+1}$ for $k = 0, \dots, 2q - 1$,
- $\mathbf{T}_k \cong \prod_{i=1}^k \widehat{\mathbf{F}_{g(i)}}$, $1 \leq k \leq q$, $1 \leq g(i) \leq f(i)$, $1 \leq i \leq q$,
- $\mathbf{T}_{q+k} \cong \prod_{i=1}^k \widehat{\mathbf{F}_{f(i)}} \times \prod_{i=k+1}^q \widehat{\mathbf{F}_{g(i)}}$, $1 \leq k \leq q - 1$,
- $\mathbf{T}_{2q} = \mathbf{T}$.

Proof. First put $\mathbf{T}_0 = \mathbf{S}$. If $q = 1$ and $s = 0$ we put $\mathbf{T}_1 = \text{Sg}^{\mathbf{T}}(\{0, d, 1\})$, d the only proper dense element of \mathbf{T} , and $\mathbf{T}_1 = \mathbf{T}_0$ if $s > 0$. Then set $\mathbf{T}_2 = \mathbf{T}$. Thus we may assume $q > 1$. If $s = 0$, that is $\mathbf{S} \cong \mathbf{2}$, let $d = 1$, else let d be the only element of $D(S) \setminus \{1\}$. Since \mathbf{S} is subdirectly irreducible

there is an $i \in \{1, \dots, q\}$ such that $(\mathbf{T})_i$ contains an isomorphic copy of \mathbf{S} and $\mathbf{S} \cong (\mathbf{S})_i$. We may assume $i = 1$, which implies

$$|(S)_i| \geq |(S)_{i+1}| \text{ for } 1 \leq i \leq q-1. \quad (1)$$

If $|(S)_i| < |S|$, $2 \leq i \leq q$, then $d_i = 1$: There are elements $a, b \in \text{Sk}(\mathbf{S})$ such that $a_i^* = b_i$ but $a_1^* \neq b_1$. Then at least one of $a_1 \wedge b_1^* > 0$ and $a_1^* \wedge b_1 > 0$ holds, thus either $a \wedge b^* = (u_1, \dots, u_q)$ or $a^* \wedge b = (u_1, \dots, u_q)$ with $u_1 > 0$ and $u_i = 0$, implying $1 = u_i^* \leq d_i$. Thus we may assume that there is $1 \leq r \leq q$ with $d = (\underbrace{e, \dots, e}_{r \text{ pl.}}, 1, \dots, 1)$. We define

$$\mathbf{S}_l \cong \begin{cases} (\mathbf{S})_l, & \text{if } (d)_l = e; \\ (\widehat{\mathbf{S}})_l, & \text{if } (d)_l = 1 \end{cases} \quad (2)$$

for $l = 1, \dots, q$. Again we consider the cases $s = 0$ and $s > 0$. If $s = 0$ we put $\mathbf{T}_1 = \text{Sg}^{\mathbf{T}}(\{0, (e, 1, \dots, 1), 1\})$, if $s > 0$ we put $\mathbf{T}_1 = \mathbf{T}_0$. Next we extend \mathbf{T}_1 to a subalgebra \mathbf{T}_2 of \mathbf{T} that is isomorphic to $(\mathbf{S})_1 \times \mathbf{S}_2$, \mathbf{S}_2 as in (2). We distinguish the cases 1. $r = 1$, that is $d = (e, 1, \dots, 1)$, and 2. $r \geq 2$.

1. In this case we have $\mathbf{S}_2 = (\widehat{\mathbf{S}})_2$, that is $|\text{D}((S)_2)| = 1$. We set $d_1 = (1, e, 1, \dots, 1)$ and $b = (1, 0, 1, \dots, 1)$. Then $\mathbf{T}_2 := \text{Sg}^{\mathbf{T}}(S \cup \{d_1, b\})$ is isomorphic to $(\mathbf{S})_1 \times (\widehat{\mathbf{S}})_2$ as $\varphi : \mathbf{T}_2 \rightarrow (\mathbf{S})_1 \times (\widehat{\mathbf{S}})_2$ defined by $\varphi(x_1, \dots, x_q) = (x_1, x_2)$ is an isomorphism: Obviously, φ is a homomorphism. The surjectivity of φ follows from $(\{b \wedge s : s \in S\})_1 \cong (\mathbf{S})_1$ and $(\{b^* \wedge s : s \in S\})_2 \cong (\mathbf{S})_2$ and $d_1 \in T_2$. The injectivity follows from (1) and the choice of b and d_1 .
2. In this case we have $\mathbf{S}_2 = (\mathbf{S})_2$, that is $|\text{D}((S)_2)| = 2$. We set $d_{1,1} = (1, e, 1, \dots, 1)$, $d_{1,2} = (e, 1, \dots, 1)$ and $b = (1, 0, 1, \dots, 1)$. Then $\mathbf{T}_2 := \text{Sg}^{\mathbf{T}}(S \cup \{d_{1,1}, d_{1,2}, b\})$ is isomorphic to $\mathbf{S} \times \mathbf{S}_2$, which is shown as in 1..

Now we show that a subalgebra $\mathbf{T}_{k-1} \cong (\mathbf{S})_1 \times \prod_{l=2}^{k-1} \mathbf{S}_l$ of \mathbf{T} can be extended to a subalgebra $\mathbf{T}_k \cong (\mathbf{S})_1 \times \prod_{l=2}^k \mathbf{S}_l$, $3 \leq k \leq q$. Under our assumption we have $(\mathbf{T}_{k-1})_{c_{k-1}} \cong \prod_{l=1}^{k-1} \mathbf{S}_l$, where $c_j := (1, \dots, 1, 0, \dots, 0)$ with the first j places equal to 0 and $(\text{D}(T_k))_k = (\text{D}(S))_k$. Here we need consider two cases, as both for $(d)_k = 1$ and $(d)_k = e$ we have $d_k, b_k \notin T_{k-1}$ for $d_k := (1, \dots, 1, e, 1, \dots, 1)$ with the k -th place equal to e .

We define $\mathbf{T}_k = \text{Sg}^{\mathbf{T}}(T_{k-1} \cup \{d_k, b_k\})$ being isomorphic to $(\mathbf{T}_{k-1})_{c_{k-1}} \times (\widehat{\mathbf{S}})_k$ as $\varphi : \mathbf{T}_k \rightarrow (\mathbf{T}_{k-1})_{c_{k-1}} \times (\widehat{\mathbf{S}})_k$ defined by $\varphi(x_1, \dots, x_q) = (x_1, \dots, x_k)$ is an isomorphism: Obviously, φ is a homomorphism. The surjectivity of φ follows from $(\{b_k \wedge s : s \in S\})_{c_{k-1}} \cong \mathbf{T}_{k-1}$ and $(\{b_k^* \wedge s : s \in S\})_k \cong (\mathbf{S})_k$ and $d_k \in \mathbf{T}_k$. Again, the injectivity follows from (1) and the choice of b_k and d_k .

After $q-1$ steps we obtain the subalgebra \mathbf{T}_q , which is isomorphic to $\prod_{l=1}^q \mathbf{S}_l$. If $|S_1| < |F_{f(1)}|$, there is $b \in \text{Sk}(\mathbf{T}_q)$ such that $b < (e, 1, \dots, 1)$ and b an anti-atom of $\text{Sk}(\mathbf{T}_q)$ but no anti-atom of $\text{Sk}(\mathbf{T})$. There is a skeletal element \bar{b} with $b < \bar{b} < (e, 1, \dots, 1)$ and $b \dot{\vee} \bar{b}^* < d$. Setting $\mathbf{T}_{q,1} = \text{Sg}^{\mathbf{T}}(T_q \cup \{\bar{b}\})$ we obtain according to [8]

$$\mathbf{T}_{q,1} = \{((\bar{b} \wedge s) \dot{\vee} (\bar{b}^* \wedge t)) \wedge d : s, t \in \text{Sk}(T_q), d \in \text{D}(T_q)\} \cong \widehat{\mathbf{F}_{r_1+1}} \times \prod_{l=2}^q \widehat{\mathbf{S}}_l, \quad (3)$$

if $r_1 \in \mathbb{N}$ is such that $\mathbf{S}_1 \cong \widehat{\mathbf{F}_{r_1}}$. Repeating this procedure for $\mathbf{T}_{q,m}$ as long as $r_1 + m < f(1)$ yields a subalgebra \mathbf{T}_{q+1} of \mathbf{T} isomorphic to $\widehat{\mathbf{F}_{f(1)}} \times \prod_{l=2}^q \mathbf{S}_l$. Applying this procedure to the factors \mathbf{S}_l for $l = 2, \dots, q$ finally finishes the proof. \square

Lemma 4.5. *If $\mathbf{T} = \mathbf{2}^p \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ with $p, q, f(i) \in \mathbb{N} \setminus \{0\}, 1 \leq i \leq q$ and $S \subseteq T$ a subalgebra isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, then there is a sequence of subalgebras $\mathbf{T}_0, \dots, \mathbf{T}_p$ of \mathbf{T} with the following properties:*

- \mathbf{T}_k is a subalgebra of \mathbf{T}_{k+1} for $k = 0, \dots, p-1$,
- $\mathbf{S} = \mathbf{T}_0$, $\mathbf{T}_k \cong \mathbf{2}^k \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ for $k = 0, \dots, p$.

Proof. As $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ there is for every $x \in S \setminus \{1\}$ a maximal dense element $d_x \neq 1$ with $x \leq d_x$. Therefore we have

$$S \cap \{x \in T : (x)_{p+i} = 1 \text{ for } i = 1, \dots, q\} = \emptyset, \quad (4)$$

thus

$$b_k := (\underbrace{1, \dots, 1}_{k \text{ pl.}}, \underbrace{0, \dots, 0}_{p-k \text{ pl.}}, 1, \dots, 1) \notin S, \quad 0 \leq k \leq p-1. \quad (5)$$

From (4) and (5) it follows that we can set $\mathbf{T}_0 = \mathbf{S}$ and $\mathbf{T}_{k+1} = \text{Sg}^{\mathbf{T}}(T_k \cup \{b_{k+1}\})$ for $k = 0, \dots, p-1$. \square

Lemma 4.6. *Let \mathbf{P} and \mathbf{Q} be p -semilattices, \mathbf{Q} an extension of \mathbf{P} , let \mathbf{S} be a finite subalgebra of \mathbf{P} with $D(\mathbf{S}) \setminus \{1\} \neq \emptyset$, and let p, q and $f(i) \geq 1, 1 \leq i \leq q$, be natural numbers. Furthermore, we may assume that \mathbf{T} is a finite subalgebra of \mathbf{Q} that is an extension of \mathbf{S} . If \mathbf{P} satisfies (AC1)–(AC4) and (EC1)–(EC5), then we have:*

1. *If $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ and $\mathbf{T} \cong \prod_{i=1}^{q-1} \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_{f(q)+1}}$, then there is an extension \mathbf{S}' of \mathbf{S} in \mathbf{P} that is isomorphic to \mathbf{T} over S .*
2. *If $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ and $\mathbf{T} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_l}$, $l \in \mathbb{N}$, then there is an extension \mathbf{S}' of \mathbf{S} in \mathbf{P} that is isomorphic to \mathbf{T} over S .*
3. *If $\mathbf{S} \cong \mathbf{2}^p \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ and $\mathbf{T} \cong \mathbf{2}^{p+1} \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, then there is an extension \mathbf{S}' of \mathbf{S} in \mathbf{P} that is isomorphic to \mathbf{T} over S .*

Proof. The proofs of 1. is straightforward, apply (AC4). To prove 2. and 3. we determine first how \mathbf{S} is contained in \mathbf{T} and then show that there is an extension \mathbf{S}' of \mathbf{S} over S isomorphic to \mathbf{T} .

1. In the case $\mathbf{T} \cong \mathbf{3}$ apply (EC3), in the case $\mathbf{T} \cong \mathbf{2}^n$ apply (EC1) $n-1$ times to obtain \mathbf{S}' .
2. There are uniquely determined $d \in D(\mathbf{S}) \setminus \{1\}$, d an anti-atom, and $b_1 \in \text{Sk}(\mathbf{S})$ such that $b_1 < d$ and b_1 is an anti-atom of $\text{Sk}(\mathbf{S})$ but no anti-atom of $\text{Sk}(\mathbf{T})$. Applying (AC4) to b_1 and d yields a skeletal element b_2 with $b_1 < b_2 < d$ and $(b_2 \wedge b_1^*)^* < d$. Putting $\mathbf{S}' = \text{Sg}^{\mathbf{P}}(S \cup \{b_2\})$ we obtain according to [8] the following.

$$\mathbf{S}' = \{((s \wedge b_2) \dot{\vee} (t \wedge b_2^*)) \wedge d : s, t \in \text{Sk}(\mathbf{S}), d \in D(\mathbf{S})\} \cong \prod_{i=1}^{q-1} \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_{f(q)+1}} \quad (6)$$

Since $\mathbf{T} \cong \prod_{i=1}^{q-1} \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_{f(q)+1}}$ there is a skeletal anti-atom $\bar{b} \in T \setminus S$ with $b_1 < \bar{b} < d$ and $(\bar{b} \wedge b_1^*)^* < d$.

Now there is according to (6) a unique isomorphism $h : \mathbf{S}' \rightarrow \mathbf{T}$ over S defined by $h(((s \wedge b_2) \dot{\vee} (t \wedge b_2^*)) \wedge d) = ((s \wedge \bar{b}) \dot{\vee} (t \wedge \bar{b}^*)) \wedge d$.

3. Since $\mathbf{T} \cong \prod_{i=1}^{q+1} \widehat{\mathbf{F}_{f(i)}}$ we may assume $\mathbf{T} = \prod_{i=1}^{q+1} \widehat{\mathbf{F}_{f(i)}}$ identifying the subalgebra \mathbf{T} of \mathbf{Q} with the direct product \mathbf{T} is isomorphic to. Furthermore, we may assume $(\mathbf{T})_i \not\cong \mathbf{2}$, $i = 1, \dots, q+1$, because the occurrence of factors $\mathbf{2}$ in \mathbf{T} is treated in 3. below.

To simplify notation we define $\vec{x} = (x_1, \dots, x_q)$ for $x \in T$, $\vec{x} \leq \vec{y}$ if $x, y \in T$ and $(x)_i \leq (y)_i$ for $1 \leq i \leq q$, and $\vec{x} < \vec{y}$ if $\vec{x} \leq \vec{y}$ and $(x)_k < (y)_k$ for a $k \in \{1, \dots, q\}$. Furthermore, we set $\vec{U} = \{\vec{x} : x \in U\}$ if U is a subset of T .

Since \mathbf{S} is isomorphic to the direct product of the subdirectly irreducible factors $\widehat{\mathbf{F}_{f(i)}}$, $i = 1, \dots, q$, and $\mathbf{T} = \prod_{i=1}^{q+1} \widehat{\mathbf{F}_{f(i)}}$ is an extension of \mathbf{S} we have — changing the enumeration if necessary — $\vec{S} = \vec{T}$, which implies $(S)_i = (T)_i$ for $i = 1, \dots, q$.

Denoting the proper dense element of $\widehat{\mathbf{F}_{f(i)}}$ by e , $i = 1, \dots, q+1$, we have $d_0 := \min(D(T)) = (e, \dots, e)$. We consider the cases (a) $\widehat{\mathbf{F}_{f(q+1)}} = \mathbf{3}$ and (b) $\widehat{\mathbf{F}_{f(q+1)}} = \widehat{\mathbf{F}_l}$, $l > 1$:

- (a) Here, we have $\mathbf{T} = \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \times \mathbf{3}$. We distinguish the subcases $\min(D(\mathbf{S}))_{q+1} = 1$ and $\min(D(\mathbf{S}))_{q+1} = e$.

In the first subcase let $a_{i,j}$, $1 \leq j \leq f(i)$, be the atoms of $\widehat{\mathbf{F}_{f(i)}}$, $1 \leq i \leq q$. Then exactly the subsets $S_{a_{i,j}}$ of T with

$$S_{a_{i,j}} := \{x \in T : ((x)_i \geq a_{i,j} \longrightarrow (x)_{q+1} = 1) \ \& \ ((x)_i \not\geq a_{i,j} \longrightarrow (x)_{q+1} = 0)\}, \quad (7)$$

, $1 \leq i \leq q$, $1 \leq j \leq f(i)$ fixed, are the subalgebras of \mathbf{T} isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$.

The subsets $S_{a_{i,j}}$ of T are easily checked to be subalgebras isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$.

For the converse let \mathbf{S} be a subalgebra of \mathbf{T} isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$. We set $a = \bigwedge \{x \in S : (x)_{q+1} = 1\} \in S$. Thus $a = (a_1, \dots, a_q, 1)$, where $a \neq (0, \dots, 0, 1)$: $(0, \dots, 0, 1) \in S$ together with $\vec{S} = \vec{T}$ would imply $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \times \mathbf{2}$. We want to show $S = S_a$, where

$$S_a := \{x \in T : (\vec{x} \geq \vec{a} \longrightarrow (x)_{q+1} = 1) \ \& \ (\vec{x} \not\geq \vec{a} \longrightarrow (x)_{q+1} = 0)\}. \quad (8)$$

$S \subseteq S_a$: For $x \in S$ we have that $\vec{x} \geq \vec{a}$ implies $(x)_{q+1} = 1$. Otherwise there would be $y \in S$ with $\vec{y} > \vec{a}$ and $(y)_{q+1} \in \{0, e\}$ (for $\vec{y} = \vec{a}$ we have $(y)_{q+1} = 1$). We show that such a y is impossible by considering (i) $(y)_{q+1} = e$ and (ii) $(y)_{q+1} = 0$. Assume (i). There is $d_y \in S$ such that $y = d_y \wedge y^{**}$. Thus $(d_y)_{q+1} = e$ which contradicts our assumption $\min(D(\mathbf{S}))_{q+1} = 1$.

Assume (ii). If $\vec{y}^* \geq \vec{a}$ then together with $\vec{y} > \vec{a}$ we obtain $\vec{a} = \vec{0}$, which yields again the contradiction $a = (0, \dots, 0, 1)$.

Otherwise $\vec{y}^* \not\geq \vec{a}$ which together with $(y^*)_{q+1} = 1$ implies $\bigwedge\{x \in S : (x)_{q+1} = 1\} < \vec{a}$ as $y^* \in S$ contradicting the definition of a .

We also have that $\vec{x} \not\geq \vec{a}$ implies $(x)_{q+1} = 0$ for $x \in S$. Else there would be $y \in S$ with $\vec{y} \geq \vec{a}$ and $(y)_{q+1} \in \{e, 1\}$. As above $(y)_{q+1} = e$ is impossible, thus $(y)_{q+1} = 1$. Again we would obtain $\bigwedge\{x \in S : (x)_{q+1} = 1\} < \vec{a}$.

$\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, $S \subseteq S_a$ and $|S_a| \leq \left| \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \right| - x \mapsto \vec{x}$ is an injection from S_a in $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ — yields $S = S_a$.

Finally, \vec{a} is an atom of $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$, thus \mathbf{S} is as claimed in (7). Otherwise \mathbf{S}_a is not a subalgebra of \mathbf{T} : There are atoms a_1, \dots, a_k of \mathbf{T} with $\vec{a} = \vec{a}_1 \dot{\vee} \dots \dot{\vee} \vec{a}_k$, $k \geq 2$. Because $\vec{a}_i < \vec{a}$ the definition of S_a yields $(\vec{a}_i, 0) \in S_a$, $i = 1, \dots, k$. We obtain

$$\begin{aligned} (a_1, \dots, a_q, 0) &= (\vec{a}, 0) \\ &= (\vec{a}_1 \dot{\vee} \dots \dot{\vee} \vec{a}_k, 0) \\ &= (\vec{a}_1, 0) \dot{\vee} \dots \dot{\vee} (\vec{a}_k, 0) \\ &\in S_a, \end{aligned}$$

which together with $(a_1, \dots, a_q, 1) \in S_a$ yields

$$(0, \dots, 0, 1) = (a_1^*, \dots, a_q^*, 1) \wedge (a_1, \dots, a_q, 1) \in S_a,$$

which we showed to be impossible.

Now, a is the only atom of \mathbf{S} such that $a \parallel d_0$. Furthermore, $a^* \wedge d_0 = a^* \wedge d_1$, $d_1 := \min(D(\mathbf{S})) = (e, \dots, e, 1) > d_0$. Applying axiom (EC5) to d_1 and a yields a dense element \check{d}_0 such that $a \parallel \check{d}_0$ and $a^* \wedge \check{d}_0 = a^* \wedge d_1$. Therefore $h_1 : \mathbf{S}' := \text{Sg}^{\mathbf{P}}(S \cup \{\check{d}_0\}) \rightarrow \mathbf{T}$ defined by

$$h_1(s) = \begin{cases} s, & s \in S; \\ d_0, & s = \check{d}_0 \end{cases}$$

is an embedding over S . We have to show that there is an extension $\tilde{\mathbf{S}}$ of \mathbf{S} such that there is an isomorphism $h : \tilde{\mathbf{S}} \rightarrow \mathbf{T}$ that is over S .

As \mathbf{P} satisfies (AC1)–(AC4) \mathbf{S}' can be extended in \mathbf{P} to a subalgebra $\mathbf{S}'' \cong \mathbf{T}$. There is a maximal dense element $d \in S'' \setminus S$. For $\mathbf{S}_1 := \text{Sg}^{\mathbf{P}}(S \cup \{d\})$ we have $D(\mathbf{S}_1) \cong D(T)$ and that there is an embedding $h_2 : \mathbf{S}_1 \rightarrow \mathbf{T}$ extending h_1 .

In the second subcase there is $k \in \{1, \dots, q\}$ with $\widehat{\mathbf{F}_{f(k)}} \cong \mathbf{3}$ because in this subcase the direct product \mathbf{S} contains a factor $\mathbf{3}$ and $\vec{S} = \vec{T}$. Thus $\min(D(\mathbf{S})) = (e, \dots, e)$, and therefore $(x)_k = (x)_{q+1}$ for $x \in S$: The last equality follows from $(\mathbf{S})_{q+1} = (\mathbf{T})_{q+1} = \mathbf{3}$, \mathbf{S} consisting of one factor $\mathbf{3}$ less than \mathbf{T} and $\mathbf{3}$ being subdirectly irreducible.

There is a unique $d \in D(\mathbf{S})$ being an anti-atom of \mathbf{S} but no anti-atom of \mathbf{T} , $d = (1, \dots, 1, e, e)$ if we assume $k = q$. Applying axiom (EC4) to d and 1 yields a dense element d_1 . There is a dense element $\tilde{d}_1 \in T$ with $d < \tilde{d}_1 < 1$. We define $h_1 : \mathbf{S}' := \text{Sg}^{\mathbf{P}}(S \cup \{d_1\}) \rightarrow T$ by setting

$$h_1(s) = \begin{cases} s, & s \in S; \\ \tilde{d}_1, & s = d_1. \end{cases}$$

To extend $D(S')$ in \mathbf{P} appropriately we again exploit that \mathbf{P} satisfies (AC1)–(AC4). \mathbf{S}' can be extended in \mathbf{P} to a subalgebra $\mathbf{S}'' \cong \mathbf{T}$. Therefore there is a maximal dense element $d_2 \in S''$ with $d = d_1 \wedge d_2$. For $\mathbf{S}_1 := \text{Sg}^{\mathbf{P}}(S' \cup \{d_2\})$ we have $D(\mathbf{S}_1) \cong D(T)$ and that there is an embedding $h_2 : \mathbf{S}_1 \rightarrow \mathbf{T}$ extending h_1 .

Thus in both subcases there is a subalgebra \mathbf{S}_1 of \mathbf{P} extending \mathbf{S} such that $D(\mathbf{S}_1) \cong 2^{q+1}$ and an embedding $h_2 : \mathbf{S}_1 \rightarrow \mathbf{T}$ over S .

In the first subcase there is by the construction of \mathbf{S}_1 a unique maximal dense element $d \in D(\mathbf{S}_1) \setminus S$. Proceeding as in the proof of Proposition 6.6 of [8] we find a skeletal element $k_d \in P$, as \mathbf{P} satisfies axioms (AC1)–(AC4), such that $\mathbf{S}_2 := \text{Sg}^{\mathbf{P}}(S_1 \cup \{k_d\}) \cong \mathbf{T}$. In the proof of Proposition 6.6 of [8] $a_d := k_d \dot{\vee} c_0^*$, from which here $k_d = a_d$ is implied by $c_0 = 1$. Therefore, the homomorphism $h : \mathbf{S}_2 \rightarrow \mathbf{T}$ extending h_2 by $h(k_d) := (1, \dots, 1, 0) \in T \setminus S$ is an isomorphism. As h_2 is over S so is h .

In the second subcase there are two maximal dense elements $d_1, d_2 \in D(\mathbf{S}_1) \setminus D(\mathbf{S})$. Again we find skeletal elements $k_{d_1}, k_{d_2} \in P$ such that $\mathbf{S}_2 := \text{Sg}^{\mathbf{P}}(S_1 \cup \{k_{d_1}, k_{d_2}\}) \cong \mathbf{T}$. Also as above we have $a_{d_1} = k_{d_1}$, $a_{d_2} = k_{d_2}$ for the elements defined in the proof of Proposition 6.6 in [8]. The homomorphism $h : \mathbf{S}_2 \rightarrow \mathbf{T}$ extending h_2 by $h(a_{d_1}) := (1, \dots, 1, 0, 1) \in T \setminus S$ and $h(a_{d_2}) := (1, \dots, 1, 0) \in T \setminus S$ is an isomorphism. As h_2 is over S so is h .

- (b) $\widehat{\mathbf{F}_{f(q+1)}} = \widehat{\mathbf{F}_l}$, $l > 1$, and there is no subalgebra \mathbf{T}' of \mathbf{T} isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_m}$, $m < l$, extending \mathbf{S} , which implies $(\mathbf{S})_{q+1} = \widehat{\mathbf{F}_l}$. Now, there is a $k \in \{1, \dots, q\}$ with $(\mathbf{S})_k \cong (\mathbf{S})_{q+1}$ and $(x)_k = (x)_{q+1}$ for $x \in S$: \mathbf{S} is the direct product of subdirectly irreducible factors as we assume $\mathbf{S} \cong \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$. Therefore, we would have $(\mathbf{S})_k \cong \widehat{\mathbf{F}_m}$ for a $m < l$. But exchanging k and $q+1$ would contradict the assumption that there is no subalgebra \mathbf{T}' of \mathbf{T} extending \mathbf{S} isomorphic to $\prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}} \times \widehat{\mathbf{F}_m}$, $m < l$, extending \mathbf{S} .

But this means that we can proceed as in (a) in the case where $(\mathbf{S})_k \cong (\mathbf{S})_{q+1} \cong \mathbf{3}$ and $(x)_k = (x)_{q+1}$ for $x \in S$.

4. We first consider the case $p = 0$ and assume $q > 0$, that is $\mathbf{T} \cong \mathbf{2} \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$. Again we assume $\mathbf{T} = \prod_{i=0}^q \widehat{\mathbf{F}_{f(i)}}$, $\widehat{\mathbf{F}_{f(0)}} := \widehat{\mathbf{F}_0} = \mathbf{2}$, identifying the subalgebra \mathbf{T} of \mathbf{Q} with the direct product \mathbf{T} is isomorphic to. Proceeding as in the proof of 3.(a) yields

$$S = S_{a_{i,j}} \quad (9)$$

where

$$S_{a_{i,j}} := \{x \in T : ((x)_i \geq a_{i,j} \longrightarrow (x)_0 = 1) \ \& \ ((x)_i \not\geq a_{i,j} \longrightarrow (x)_0 = 0)\}, \quad (10)$$

$a_{i,j}$ an atom of $\widehat{\mathbf{F}_{f(i)}}$ for an $i \in \{1, \dots, q\}$ and a $j \in \{1, \dots, f(i)\}$. For $\bar{b} := (0, 1, \dots, 1) \in T \setminus S$ we have $\bar{b} \parallel d$ and $\bar{b}^* < d$ for all

$d \in D(T) \setminus \{1\}$. We obtain

$$T = S \cup \{d \wedge \bar{b} \wedge s : d \in D(\mathbf{S}), s \in \text{Sk}(\mathbf{S}), (s)_0 = 1\} \cup \\ \{d \wedge (\bar{b} \wedge s)^* : d \in D(\mathbf{S}), s \in \text{Sk}(\mathbf{S}), (s)_0 = 1\} \quad (11)$$

as follows: From (9) and (10) it follows

$$T \setminus S = \{x \in T : ((x)_i \geq a_{i,j} \longrightarrow (x)_0 = 0) \ \& \\ ((x)_i \not\geq a_{i,j} \longrightarrow (x)_0 = 1)\}. \quad (12)$$

Let $x \in T \setminus S$ be such that $(x)_i \not\geq a_{i,j}$ and $(x)_0 = 1$. There is a $d_x \in D(\mathbf{T}) = D(\mathbf{S})$ with $x = d_x \wedge x^{**}$. For $t := x^{**}$ due to (10), as $t \notin S$ follows from $x \notin S$, we have $(t)_0 = 1$ and $(t)_i \not\geq a_{i,j}$. For $u \in T$ such that $(u)_0 = 0$ and $(u)_k = (t)_k$ for $k = 1, \dots, q$ we have $u \in \text{Sk}(\mathbf{S})$ according to (10). Setting $s = u^*$ we obtain $t = \bar{b}^* \dot{\vee} u = (\bar{b} \wedge u^*)^* = (\bar{b} \wedge s)^*$, thus $x = d_x \wedge t = d_x \wedge (\bar{b} \wedge s)^*$ with $s \in S$ and $(s)_0 = 1$. Similarly one shows that for $x \in T \setminus S$ such that $(x)_i \geq a_{i,j}$ and $(x)_0 = 0$ there is $s \in \text{Sk}(\mathbf{S})$ with $(s)_0 = 1$ and $d \in D(\mathbf{S})$ with $x = d \wedge s \wedge \bar{b}$. Obviously, the right hand side of (11) is a disjoint union.

Now we are going to show that there is a skeletal element $b \in P$ that behaves with respect to \mathbf{S} in the same way as \bar{b} . In order to express what this means, we define a_{d_m} to be the maximal central element below the maximal dense element d_m , $1 \leq m \leq q$. Therefore, $(d_m)_k = e$ if and only if $m = k$, and

$$(a_{d_m})_k = \begin{cases} 1, & k \neq m; \\ 0, & k = m \end{cases} \quad (m \neq i), \quad (a_{d_i})_k = \begin{cases} 1, & k \notin \{0, i\}; \\ 0; & k \in \{0, i\}. \end{cases}$$

Furthermore, we have

$$a_{d_i} = \dot{\bigvee} \{a_{d_m}^* : 1 \leq m \leq q, m \neq i\} \quad (13)$$

$$\bar{b} \parallel d_m \ \& \ \bar{b}^* < a_{d_m} \text{ for } m \in \{1, \dots, q\} \setminus \{i\} \quad (14)$$

$$a_{d_i} < \bar{b} \ \& \ \bar{b} \parallel d_i \ \& \ (\bar{b} \wedge a_{d_i}^*)^* < d_i \ \& \ \bar{b} \wedge a_{d_i}^* \parallel d_i \quad (15)$$

We may assume $i = q$ in (10). Let b_q be the result of applying (EC2) to a_{d_q} and d_q . Then (14) and (15) are satisfied if \bar{b} is substituted by b_q : We have $a_{d_q} \stackrel{(13)}{=} (\bigwedge \{a_{d_m} : 1 \leq m \leq q-1\})^* = \dot{\bigvee} \{a_{d_m}^* : 1 \leq m \leq q-1\} < b_q$, which implies (14). b_q satisfies (15), as b_q is obtained by applying (EC2) to a_{d_q} and d_q .

Furthermore, by setting $b = b \dot{\vee} s_0$, $s_0 = \dot{\bigvee} \{s \in \text{Sk}(\mathbf{S}) : (s)_0 = 0\}$, (14) and (15) remain valid, if b_q is replaced by b , and we additionally have

$$(\forall s \in S)((s)_0 = 0 \longrightarrow s < b). \quad (16)$$

Observe, that $b \neq 1$ holds: Otherwise $b_q^* \wedge s_0^* = 0$, thus $b_q \geq s_0^*$. As $b_q \geq a_{d_m}^*$ for $1 \leq q-1$, $b_q > a_{d_q}$ by the definition of b_q and since $s_0^* \dot{\vee} a_{d_1}^* \dot{\vee} \dots \dot{\vee} a_{d_{q-1}}^* \dot{\vee} a_{d_q} = 1$, we would obtain $b_q = 1$.

Now we show that for $\mathbf{S}' := \text{Sg}^{\mathbf{P}}(S \cup \{b\})$ there is an isomorphism $h : \mathbf{T} \rightarrow \mathbf{S}'$ over S with $h(\bar{b}) := b$: As rhs(11) is a disjoint union

$$h(x) := \begin{cases} x, & x \in S; \\ d \wedge b \wedge s, & x = d \wedge \bar{b} \wedge s, s \in \text{Sk}(\mathbf{S}), (s)_0 = 1, \\ & d \in \text{D}(\mathbf{S}); \\ d \wedge (b \wedge s)^*, & x = d \wedge (\bar{b} \wedge s)^*, s \in \text{Sk}(\mathbf{S}), (s)_0 = 1, \\ & d \in \text{D}(\mathbf{S}) \end{cases}$$

is well-defined. Obviously, h is over S .

We have to show that for all $u, v \in T$

$$h(u \wedge v) = h(u) \wedge h(v) \quad (17)$$

$$h(u^*) = h(u)^* \quad (18)$$

hold and that h is bijective.

For (17) we consider, assuming $(s_1)_0 = (s_2)_0 = 1$, the following cases:

- $u = d_1 \wedge (\bar{b} \wedge s_1)^*, v = d_2 \wedge (\bar{b} \wedge s_2)^*$.

$$\begin{aligned} h(u \wedge v) &= h(d_1 \wedge (\bar{b} \wedge s_1)^* \wedge d_2 \wedge (\bar{b} \wedge s_2)^*) \\ &= h(d_1 \wedge d_2 \wedge ((\bar{b} \wedge s_1) \dot{\vee} (\bar{b} \wedge s_2))^*) \\ &= h(d_1 \wedge d_2 \wedge (\bar{b} \wedge (s_1 \dot{\vee} s_2))^*) \\ &= d_1 \wedge d_2 \wedge (b \wedge (s_1 \dot{\vee} s_2))^* \\ &= d_1 \wedge d_2 \wedge ((b \wedge s_1) \dot{\vee} (b \wedge s_2))^* \\ &= d_1 \wedge d_2 \wedge (b \wedge s_1)^* \wedge (b \wedge s_2)^* \\ &= h(u) \wedge h(v) \end{aligned}$$

- $u = d_1 \wedge \bar{b} \wedge s_1, v = d_2 \wedge (\bar{b} \wedge s_2)^*$.

$$\begin{aligned} h(u \wedge v) &= h(d_1 \wedge \bar{b} \wedge s_1 \wedge d_2 \wedge (\bar{b} \wedge s_2)^*) \\ &= h(d_1 \wedge \bar{b} \wedge s_1 \wedge d_2 \wedge (\bar{b}^* \dot{\vee} s_2^*)) \\ &= h(d_1 \wedge d_2 \wedge s_1 \wedge ((\bar{b} \wedge \bar{b}^*) \dot{\vee} (\bar{b} \wedge s_2^*))) \\ &= h(d_1 \wedge d_2 \wedge s_1 \wedge \bar{b} \wedge s_2^*) \\ &\stackrel{\bar{b} > s_2^*}{=} h(d_1 \wedge d_2 \wedge s_1 \wedge s_2^*) \\ &= d_1 \wedge d_2 \wedge s_1 \wedge s_2^* \\ &\stackrel{(16)}{=} d_1 \wedge d_2 \wedge s_1 \wedge (b \wedge (b^* \dot{\vee} s_2^*)) \\ &= d_1 \wedge b \wedge s_1 \wedge d_2 \wedge (b \wedge s_2)^* \\ &= h(u) \wedge h(v) \end{aligned}$$

- $u \in S, v = d \wedge \bar{b} \wedge s$ with $(s)_0 = 1$. We consider two subcases:
 - $(u)_0 = 1$. Then $(u \wedge s)_0 = 1$, thus

$$\begin{aligned} h(u \wedge v) &= h(u \wedge d \wedge \bar{b} \wedge s) \\ &= h(d \wedge \bar{b} \wedge (u \wedge s)) \\ &= d \wedge b \wedge (u \wedge s) \\ &= u \wedge (d \wedge b \wedge s) \\ &= h(u) \wedge h(v) \end{aligned}$$

$$- (u)_0 = 0.$$

$$\begin{aligned}
h(u \wedge v) &= h(u \wedge d \wedge \bar{b} \wedge s) \\
&\stackrel{\bar{b} \geq u}{=} h(d \wedge u \wedge s) \\
&= d \wedge u \wedge s \\
&\stackrel{(16)}{=} u \wedge d \wedge b \wedge s \\
&= h(u) \wedge h(v)
\end{aligned}$$

- $u \in S, v = d \wedge (\bar{b} \wedge s)^*$ with $(s)_0 = 1$. There is $d_u \in D(\mathbf{S})$ with $u = d_u \wedge u^{**}$. We consider again two subcases:
 - $(u)_0 = 1$. Then $(u \wedge s)_0 = 1$, thus

$$\begin{aligned}
h(u \wedge v) &= h(d_u \wedge u^{**} \wedge d \wedge (\bar{b} \wedge s)^*) \\
&= h(d_u \wedge d \wedge (u^* \dot{\vee} (\bar{b} \wedge s))^*) \\
&= h(d_u \wedge d \wedge ((u^* \dot{\vee} \bar{b}) \wedge (u^* \dot{\vee} s))^*) \\
&\stackrel{\bar{b} \geq u^*}{=} h(d_u \wedge d \wedge (\bar{b} \wedge (u^* \dot{\vee} s))^*) \\
&= d_u \wedge d \wedge (b \wedge (u^* \dot{\vee} s))^* \\
&\stackrel{(16)}{=} d_u \wedge d \wedge ((u^* \dot{\vee} b) \wedge (u^* \dot{\vee} s))^* \\
&= d_u \wedge d \wedge (u^* \dot{\vee} (b \wedge s))^* \\
&= d_u \wedge u^{**} \wedge d \wedge (b \wedge s)^* \\
&= h(u) \wedge h(v)
\end{aligned}$$

$$- (u)_0 = 0.$$

$$\begin{aligned}
h(u \wedge v) &= h(u \wedge d \wedge (\bar{b} \wedge s)^*) \\
&= h(u \wedge d \wedge (\bar{b}^* \dot{\vee} s^*)) \\
&= h(d \wedge ((u \wedge \bar{b}^*) \dot{\vee} (u \wedge s^*))) \\
&\stackrel{\bar{b} \geq u}{=} h(d \wedge u \wedge s^*) \\
&= d \wedge u \wedge s^* \\
&\stackrel{(16)}{=} d \wedge ((u \wedge b^*) \dot{\vee} (u \wedge s^*)) \\
&= u \wedge d \wedge (b^* \dot{\vee} s^*) \\
&= u \wedge d \wedge (b \wedge s)^* \\
&= h(u) \wedge h(v)
\end{aligned}$$

For (18) we consider, assuming $(s)_0 = 1$, the following cases:

- $u = d \wedge \bar{b} \wedge s$.

$$\begin{aligned}
h(u^*) &= h\left((d \wedge \bar{b} \wedge s)^*\right) \\
&= h\left((d \wedge \bar{b} \wedge s)^{***}\right) \\
&= h\left(\left(d^{**} \wedge (\bar{b} \wedge s)^{**}\right)^*\right) \\
&= h\left(1 \wedge (\bar{b} \wedge s)^*\right) \\
&= 1 \wedge (b \wedge s)^* \\
&= (d \wedge b \wedge s)^* \\
&= h(u)^*
\end{aligned}$$

- $u = d \wedge (\bar{b} \wedge s)^*$.

$$\begin{aligned}
h(u^*) &= h\left(\left(d \wedge (\bar{b} \wedge s)^*\right)^*\right) \\
&= h\left(1 \wedge (\bar{b} \wedge s)^{**}\right) \\
&= h(1 \wedge \bar{b} \wedge s) \\
&= 1 \wedge b \wedge s \\
&= (d \wedge (b \wedge s)^*)^* \\
&= h\left(d \wedge (\bar{b} \wedge s)^*\right)^* \\
&= h(u)^*
\end{aligned}$$

It remains to show that h is bijective. For injectivity let $x, y \in T$ with $x \neq y$. If $x, y \in S$ then $h(x) \neq h(y)$ trivially holds. We consider the following non-trivial cases:

- $x \in S, y \in T \setminus S$. We consider the following subcases:
 - $(x)_0 = 1, y = d_y \wedge \bar{b} \wedge s_y, s_y \in \text{Sk}(\mathbf{S})$. Then $h(x) = h(y)$ is impossible:

$$\begin{aligned}
h(x) = h(y) &\implies x = d_y \wedge b \wedge s_y \\
&\implies x^{**} = b \wedge s_y \\
&\implies a_{d_q}^* \dot{\vee} x^{**} = a_{d_q}^* \dot{\vee} (b \wedge s_y) \\
&\implies a_{d_q}^* \dot{\vee} x^{**} = \left(a_{d_q}^* \dot{\vee} b\right) \wedge \left(a_{d_q}^* \dot{\vee} s_y\right) \\
&\stackrel{(15)}{\implies} a_{d_q}^* \dot{\vee} x^{**} = a_{d_q}^* \dot{\vee} s_y \\
&\implies \left(a_{d_q}^* \dot{\vee} x^{**}\right)_q = \left(a_{d_q}^* \dot{\vee} s_y\right)_q \\
&\implies (x^{**})_q = (s_y)_q.
\end{aligned}$$

But the last equation contradicts $(x)_q \not\geq a_{q,j}, (s_y)_q \geq a_{q,j}$.
– $(x)_0 = 0, y = d_y \wedge \bar{b} \wedge s_y$. $h(x) = h(y)$ is impossible:

$$\begin{aligned}
h(x) = h(y) &\implies x = d_y \wedge b \wedge s_y \\
&\implies x^{**} = b \wedge s_y \\
&\implies b^* \dot{\vee} x^{**} = b^* \dot{\vee} s_y \\
&\stackrel{(14)}{\implies} a_{d_1} \dot{\vee} x^{**} = a_{d_1} \dot{\vee} s_y \\
&\implies (a_{d_1} \dot{\vee} x^{**})_q = (a_{d_1} \dot{\vee} s_y)_q \\
&\implies (x^{**})_q = (s_y)_q
\end{aligned}$$

But the last equation contradicts $(x)_q \not\geq a_{q,j}$, $(y)_q \geq a_{q,j}$.
 – $(x)_0 = 1$, $y = d_y \wedge (\bar{b} \wedge s_y)^*$. $h(x) = h(y)$ is impossible:

$$\begin{aligned}
 h(x) = h(y) &\implies x = d_y \wedge (b \wedge s_y)^* \\
 &\implies x^{**} = (b \wedge s_y)^* \\
 &\implies x^{**} = b^* \dot{\vee} s_y^* \\
 &\implies b \wedge x^{**} = b \wedge s_y^* \\
 &\implies b^* \dot{\vee} x^* = b^* \dot{\vee} s_y^* \\
 &\stackrel{(14)}{\implies} a_{d_1} \dot{\vee} x^* = a_{d_1} \dot{\vee} s_y^* \\
 &\implies (a_{d_1} \dot{\vee} x^*)_q = (a_{d_1} \dot{\vee} s_y^*)_q \\
 &\implies (x^*)_q = (s_y)_q
 \end{aligned}$$

But the last equation contradicts $(x^*)_q \not\geq a_{q,j}$, $(y)_q \geq a_{q,j}$.
 – $(x)_0 = 0$, $y = d_y \wedge (\bar{b} \wedge s_y)^*$. $h(x) = h(y)$ is impossible:

$$\begin{aligned}
 h(x) = h(y) &\implies x = d_y \wedge (b \wedge s_y)^* \\
 &\implies x^{**} = (b \wedge s_y)^* \\
 &\implies x^* = b \wedge s_y \\
 &\implies b^* \dot{\vee} x^* = b^* \dot{\vee} s_y \\
 &\stackrel{(14)}{\implies} a_{d_1} \dot{\vee} x^* = a_{d_1} \dot{\vee} s_y \\
 &\implies (a_{d_1} \dot{\vee} x^*)_0 = (a_{d_1} \dot{\vee} s_y)_0 \\
 &\implies 1 = (s_y)_q
 \end{aligned}$$

• $x, y \in T \setminus S$. We consider the following subcases:

– $x = d_x \wedge \bar{b} \wedge s_x$, $y = d_y \wedge \bar{b} \wedge s_y$. We obtain

$$\begin{aligned}
 h(x) = h(y) &\implies d_x \wedge b \wedge s_x = d_y \wedge b \wedge s_y \\
 &\implies (b \wedge s_x)^{**} = (b \wedge s_y)^{**} \\
 &\implies b \wedge s_x = b \wedge s_y \\
 &\implies b^* \dot{\vee} (b \wedge s_x) = b^* \dot{\vee} (b \wedge s_y) \\
 &\implies b^* \dot{\vee} s_x = b^* \dot{\vee} s_y \\
 &\stackrel{(16)}{\implies} s_x = s_y.
 \end{aligned}$$

If $d_x \wedge \bar{b} \wedge s_x \neq d_y \wedge \bar{b} \wedge s_y$ then because of $(\bar{b})_0 = 0$ there is, setting $s = s_x = s_y$, $m \in \{1, \dots, q\}$ with $(d_x)_m = a_{d_m}^* \wedge d_x = e$, $(d_y)_m = a_{d_m}^* \wedge d_y = (s)_m = 1$. This yields $a_{d_m}^* \wedge s \wedge d_x \neq a_{d_m}^* \wedge d_y \wedge s$ contradicting our assumption $h(x) = h(y)$.

In the case $m < q$ we have due to (14) $a_{d_m}^* < b$, thus $d_x \wedge b \wedge s_x \neq d_y \wedge b \wedge s_y$.

In the case $m = q$ we have $b \wedge a_{d_q}^* \parallel d_q$, which is (15). Furthermore, $s \geq a_{d_q}^*$ as $(s)_0 = (s)_q = 1$. We obtain $h(y) = d_y \wedge b \wedge s \parallel d_q$. On the other hand because of $d_x \leq d_q$ we have $h(x) \leq d_x \leq d_q$, again contradicting our assumption.

– $x = d_x \wedge (\bar{b} \wedge s_x)^*$, $y = d_y \wedge (\bar{b} \wedge s_y)^*$. This subcase is very similar to the preceding one:

$$\begin{aligned}
 h(x) = h(y) &\implies d_x \wedge (b \wedge s_x)^* = d_y \wedge (b \wedge s_y)^* \\
 &\implies (b \wedge s_x)^{***} = (b \wedge s_y)^{***} \\
 &\implies b \wedge s_x = b \wedge s_y
 \end{aligned}$$

- $x = d_x \wedge \bar{b} \wedge s_x, y = d_y \wedge (\bar{b} \wedge s_y)^*$. Here $h(x) = h(y)$ implies $b \wedge s_x = s_y^* \dot{\vee} b^*$ which is impossible.

The definition of h implies, that the surjectivity of h amounts to the validity of

$$S' = S \cup \{d \wedge b \wedge s : d \in D(\mathbf{S}), s \in \text{Sk}(\mathbf{S}), (s)_0 = 1\} \cup \{d \wedge (b \wedge s)^* : d \in D(\mathbf{S}), s \in \text{Sk}(\mathbf{S}), (s)_0 = 1\}. \quad (19)$$

That rhs(19) is contained in S' and that rhs(19) contains $S \cup \{b\}$ is obvious. For the converse we have to show that rhs(19) is closed under the operations. We consider the cases that are not obvious. In the sequel we assume $d \in D(\mathbf{S})$ and $s \in \text{Sk}(\mathbf{S})$ with $(s)_0 = 1$.

$$\begin{aligned} (d \wedge (b \wedge s)^*)^* &= (d \wedge (b \wedge s)^*)^{***} \\ &= (d^{**} \wedge (b \wedge s)^{***})^* \\ &= ((b \wedge s)^*)^* \\ &= b \wedge s \\ &= 1 \wedge b \wedge s, \end{aligned}$$

$$\begin{aligned} (d \wedge (b \wedge s))^* &= (d \wedge (b \wedge s))^{***} \\ &= (d^{**} \wedge (b \wedge s)^{**})^* \\ &= (b \wedge s)^* \\ &= 1 \wedge (b \wedge s)^*, \end{aligned}$$

$$\begin{aligned} d_1 \wedge (b \wedge s_1)^* \wedge d_2 \wedge (b \wedge s_2)^* &= d_1 \wedge d_2 \wedge (b^* \dot{\vee} s_1)^* \wedge (b^* \dot{\vee} s_2)^* \\ &\stackrel{d_3 := d_1 \wedge d_2}{=} d_3 \wedge (b^* \dot{\vee} (s_1^* \wedge s_2^*)) \\ &= d_3 \wedge (b \wedge (s_1^* \wedge s_2^*))^*, \end{aligned}$$

with $(s_1 \dot{\vee} s_2)_0 = 1$.

$$\begin{aligned} d_1 \wedge (b \wedge s_1)^* \wedge d_2 \wedge (b \wedge s_2) &= d_1 \wedge d_2 \wedge b^* \dot{\vee} s_1^* \wedge b \wedge s_2 \\ &\stackrel{d_3 := d_1 \wedge d_2}{=} d_3 \wedge (s_1^* \wedge b) \wedge s_2 \\ &\stackrel{(16)}{=} d_3 \wedge s_1^* \wedge s_2 \end{aligned}$$

Finally, we look at $x \in S$ and show that $x \wedge d \wedge (b \wedge s)$ and $x \wedge d \wedge (b \wedge s)^*$ are also contained in rhs(19). First we consider $x \wedge d \wedge (b \wedge s)$. If $(x)_0 = 1$ then $x \wedge d \wedge (b \wedge s)$ is contained in rhs(19) since $(x \wedge s)_0 = 1$. If $(x)_0 = 0$ then $x \wedge d \wedge (b \wedge s) \stackrel{(16)}{=} x \wedge d \wedge s \in S$. Next we consider $x \wedge d \wedge (b \wedge s)^*$. There is $d_x \in D(\mathbf{S})$ with $x = d_x \wedge x^{**}$. First we assume $(x)_0 = 0$.

$$\begin{aligned} x \wedge d \wedge (b \wedge s)^* &= d \wedge d_x \wedge x^{**} \wedge (b \wedge s)^* \\ &\stackrel{d_3 := d \wedge d_x}{=} d_3 \wedge (x^* \dot{\vee} (b \wedge s))^* \\ &= d_3 \wedge ((x^* \dot{\vee} b) \wedge (x^* \dot{\vee} s))^* \\ &\stackrel{(16), (x)_0=0}{=} d_3 \wedge (x^* \dot{\vee} s)^* \end{aligned}$$

Now let $(x)_0 = 1$.

$$\begin{aligned}
x \wedge d \wedge (b \wedge s)^* &= d \wedge d_x \wedge x^{**} \wedge (b \wedge s)^* \\
&\stackrel{d_3 := d \wedge d_x}{=} d_3 \wedge (x^* \dot{\vee} (b \wedge s))^* \\
&= d_3 \wedge ((x^* \dot{\vee} b) \wedge (x^* \dot{\vee} s))^* \\
&\stackrel{(16), (x^*)_0=0}{=} d_3 \wedge (b \wedge (x^* \dot{\vee} s))^* \\
&= d_3 \wedge (b \wedge (x^{**} \wedge s^*))^*,
\end{aligned}$$

with $(x^{**} \wedge s^*)^* \in S$ and $((x^{**} \wedge s^*)^*)_0 \geq (s^{**})_0 \geq (s)_0 = 1$.

Note that a_{d_q} is the only maximal skeletal (central) element of \mathbf{S} that is not a maximal skeletal element of \mathbf{S}' anymore. In \mathbf{S}' we have $a_{d_q} < b^* \dot{\vee} a_{d_q} = (b \wedge a_{d_q}^*)^* < d_q$.

We now consider the case $p > 0$ and assume again $q > 0$. Since $p > 0$ there is a unique anti-atom b_1 of $\text{Sk}(\mathbf{S})$ such that $b_1 \parallel d$ for all $d \in D(\mathbf{S}) \setminus \{1\}$ and b_1 is not an anti-atom of \mathbf{T} . Applying (EC1) to b_1 and 1 yields a skeletal element b_2 with $b_1 < b_2 < 1$. Since $\mathbf{T} \cong \mathbf{2}^{p+1} \times \prod_{i=1}^q \widehat{\mathbf{F}_{f(i)}}$ there is a skeletal anti-atom $\bar{b} \in T \setminus S$ with $b_1 < \bar{b} < 1$. Setting $\mathbf{S}' = \text{Sg}^{\mathbf{P}}(S \cup \{b_2\})$ there is a unique isomorphism $h : \mathbf{S}' \rightarrow \mathbf{T}$ over S and $h(b_2) = \bar{b}$:

This holds because b_2 and \bar{b} satisfy the same equations with respect to $D(\mathbf{S})$ as b_1 and because there is a unique isomorphism $h_1 : \text{Sg}^{\mathbf{P}}(\text{Sk}(\mathbf{S}) \cup \{b_2\}) \rightarrow \text{Sg}^{\mathbf{Q}}(\text{Sk}(\mathbf{S}) \cup \{\bar{b}\})$ over $\text{Sk}(\mathbf{S})$, see Remark 4.2.

□

Theorem 4.7. *A p -semilattice \mathbf{P} is existentially closed if and only if \mathbf{P} satisfies (AC1)–(AC4) and (EC1)–(EC5).*

Proof. The proof is split up in a necessity and a sufficiency part.

1. *Necessity*

The necessity of the axioms (AC1)–(AC4) follows from Theorem 3.3 because every existentially closed p -semilattice is algebraically closed.

For the necessity of the axioms (EC1)–(EC5) we consider the following \exists -sentences of $\mathcal{L}(\mathbf{P})$:

$\varphi_1(b_1, b_2)$: $(\exists x)(\text{Sk}(x) \ \& \ b_1 < x < b_2)$ with $\text{Sk}(b_1), \text{Sk}(b_2)$ and $b_1 < b_2$,

$\varphi_2(b, d)$: $(\exists x)(\text{Sk}(x) \ \& \ b < x \parallel d \ \& \ (x \wedge b^*)^* < d \ \& \ x \wedge b^* \parallel d)$ with $\text{Sk}(b), D(d)$ and $b < d \ \& \ b^* \parallel d$,

φ_3 : $(\exists x)(D(x) \ \& \ x < 1)$,

$\varphi_4(d_1, d_2)$: $(\exists x)(D(x) \ \& \ d_1 < x < d_2)$ with $D(d_1), D(d_2)$ and $d_1 < d_2$,

$\varphi_5(b, d)$: $(\exists x)(D(x) \ \& \ x < d \ \& \ x \parallel b \ \& \ x \wedge b^* = d \wedge b^*)$ with $D(d) \ \& \ \text{Sk}(b)$ and $0 < b < d$.

Each of these sentences can be satisfied in some direct product $\mathbf{P}' \supseteq \mathbf{P}$ with suitably many subdirectly irreducible factors $\mathbf{2}$ and $\widehat{\mathbf{B}_i}$, $i \in I$, \mathbf{B}_i boolean algebras.

2. *Sufficiency*

This part is an adaptation of the sufficiency part of the first part of the proof of Theorem 4.2 in [3]. Let \mathbf{P} be a p -semilattice satisfying

(AC1)-(AC4) and (EC1)-(EC5). We prove that \mathbf{P} is existentially closed by showing that for any extension \mathbf{Q} of \mathbf{P} , $a_1, \dots, a_m \in P$ and $v_1, \dots, v_n \in Q$ arbitrary, there exist $u_1, \dots, u_n \in P$ such that $\text{Sg}^{\mathbf{P}}(\{a_1, \dots, a_m, u_1, \dots, u_n\})$ and $\text{Sg}^{\mathbf{Q}}(\{a_1, \dots, a_m, v_1, \dots, v_n\})$ are isomorphic over $\{a_1, \dots, a_m\}$:

If $\mathbf{Q} \models (\exists x_1, \dots, x_n) \varphi(x_1, \dots, x_n, a_1, \dots, a_m)$ with φ a quantifier-free $\mathcal{L}(\mathbf{P})$ -formula, say $\mathbf{Q} \models \varphi(\vec{v}, \vec{a})$, then by isomorphism over $\{a_1, \dots, a_m\}$ we obtain $\mathbf{P} \models \varphi(\vec{u}, \vec{a})$, thus $\mathbf{P} \models (\exists \vec{x}) \varphi(\vec{x}, \vec{a})$. Every finite system of equations and negated equations with coefficients $a_1, \dots, a_m \in P$ corresponds to a formula $\varphi(\vec{x}, \vec{a})$.

To simplify notation we define $S = \{a_1, \dots, a_m\}$ and $T = \{a_1, \dots, a_m, v_1, \dots, v_n\}$, where we may assume that \mathbf{S} and \mathbf{T} are subalgebras of \mathbf{P} and \mathbf{Q} , respectively (otherwise consider $\text{Sg}(S)$ and $\text{Sg}(T)$).

We may assume

$$\mathbf{S} \cong \mathbf{2}^r \times \widehat{\mathbf{F}_t^s}, \quad r, s, t \in \mathbb{N} : \quad (20)$$

According to Theorem 3.3 \mathbf{P} is algebraically closed since \mathbf{P} satisfies (AC1)-(AC4). Therefore, according to Theorem 3.1, any finite subalgebra can be extended within \mathbf{P} first to a subalgebra $\mathbf{2}^r \times (\widehat{\mathbf{A}})^s$, $r, s \in \mathbb{N}$, thus to a subalgebra isomorphic to $\mathbf{2}^r \times \widehat{\mathbf{F}_t^s}$, $r, s \in \mathbb{N}$ and some suitable $t \in \mathbb{N}$.

Furthermore, using subdirect representation,

$$\mathbf{Q} = \widehat{\mathbf{B}}^I \quad (21)$$

may be assumed for a suitable atomfree boolean algebra \mathbf{B} and a suitable index set I .

Let $c_1, \dots, c_r, c_{r+1}, \dots, c_{r+s}$ be the elements of S corresponding to the $r+s$ (central) elements $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ of $\mathbf{2}^r \times \widehat{\mathbf{F}_t^s}$. We have

$$\mathbf{Q} \cong \prod_{k=1}^{r+s} \mathbf{Q}_{c_k} \quad (22)$$

with \mathbf{Q}_{c_k} still being of type (21). Furthermore,

$$\mathbf{P} \cong \prod_{k=1}^{r+s} \mathbf{P}_{c_k}, \quad (23)$$

and \mathbf{P}_{c_k} still satisfies axioms (AC1)-(AC4) and (EC1)-(EC5) by Lemma 4.3. We also have, using (20),

$$\mathbf{S}_{c_k} \cong \begin{cases} \mathbf{2}, & 1 \leq k \leq r; \\ \widehat{\mathbf{F}_t}, & r+1 \leq k \leq r+s. \end{cases} \quad (24)$$

As a direct product of algebraically closed factors the p-semilattices of type (22) \mathbf{Q} and \mathbf{Q}_{c_k} are algebraically closed according to Lemma 5 of [11]. Therefore, as above for \mathbf{S} ,

$$\{a_1, \dots, a_m, v_1, \dots, v_n\}_{c_k} \cong \mathbf{2}^{p_k} \times \prod_{i=1}^q \widehat{\mathbf{F}_{f_k(i)}} \quad (p, q, f_k(i) \in \mathbb{N}) \quad (25)$$

may be assumed.

Summing up, the preceding considerations yield: To show that for all $a_1, \dots, a_m \in P$ and $v_1, \dots, v_n \in Q$ there are $u_1, \dots, u_n \in P$ such that $\bar{S} := \{a_1, \dots, a_m, u_1, \dots, u_n\}$ and $T := \{a_1, \dots, a_m, v_1, \dots, v_n\}$ are isomorphic over $S := \{a_1, \dots, a_m\}$, due to (22)-(25)

$$\mathbf{S} \cong \widehat{\mathbf{F}}_t \quad (t \in \mathbb{N}) \quad (26)$$

with $\widehat{\mathbf{F}}_0 := \mathbf{2}$,

$$\mathbf{T} \cong \mathbf{2}^p \times \prod_{i=1}^q \widehat{\mathbf{F}}_{f(i)} \quad (p, q, f(i) \in \mathbb{N}) \quad (27)$$

and $t \leq f(1)$ may be assumed.

If $\mathbf{S} = \mathbf{2}$ and $\mathbf{T} \cong \mathbf{2}^n$ then applying (EC1) yields that there is a subalgebra $\tilde{\mathbf{S}}$ of \mathbf{P} and an isomorphism $f : \tilde{\mathbf{S}} \rightarrow \mathbf{T}$ over S . If $\mathbf{S} = \mathbf{2}$ and \mathbf{T} contains a dense element different from 1, we first extend \mathbf{S} within \mathbf{P} to $\mathbf{3}$ by applying (EC3). Therefore we assume $\mathbf{S} \cong \widehat{\mathbf{F}}_l$, $1 \leq l$ in the sequel.

According to Lemma 4.4 there is a sequence $\mathbf{T}_0, \dots, \mathbf{T}_{2q}$ of subalgebras of \mathbf{Q} with $\mathbf{T}_0 = \mathbf{S}$ and $\mathbf{T}_{2q} \cong \prod_{i=1}^q \widehat{\mathbf{F}}_{f(i)}$ such that for $i = 1, \dots, 2q - 1$ we have $\mathbf{T}_i \leq \mathbf{T}_{i+1}$, whereby

$$\mathbf{T}_{i+1} \cong \mathbf{T}_i \times \widehat{\mathbf{F}}_{l_i} \quad i = 1, \dots, q - 1, \quad l_i \leq f(i + 1), \quad (28)$$

$$\mathbf{T}_{q+i} \cong \prod_{j=1}^i \widehat{\mathbf{F}}_{f(j)} \times \prod_{j=i+1}^q \widehat{\mathbf{F}}_{l_j} \quad i = 1, \dots, q. \quad (29)$$

In (29) there is for every $i \in \{1, \dots, q\}$ a sequence $\mathbf{T}_{i,0}, \dots, \mathbf{T}_{i,f(i)-l_i}$ such that for $0 < j < f(i) - l_i$

$$\mathbf{T}_{i,j} \leq \mathbf{T}_{i,j+1}, \quad \mathbf{T}_{i,j} \cong \prod_{k=1}^{i-1} \widehat{\mathbf{F}}_{f(k)} \times \widehat{\mathbf{F}}_{l_i+j} \times \prod_{k=i+1}^q \widehat{\mathbf{F}}_{l_k}. \quad (30)$$

Finally, there is according to Lemma 4.5 a sequence $\mathbf{U}_0, \dots, \mathbf{U}_p$ of subalgebras of \mathbf{Q} with

$$\mathbf{U}_j \cong \mathbf{2}^j \times \prod_{i=1}^q \widehat{\mathbf{F}}_{f(i)}, \quad j = 0, \dots, p. \quad (31)$$

According to Lemma 4.6,2. there exists for every $i \in \{1, \dots, q - 1\}$ a subalgebra \mathbf{S}_{i+1} of \mathbf{P} and an isomorphism $f_i : \mathbf{S}_{i+1} \rightarrow \mathbf{T}_{i+1}$ over S_i , the sequence $(\mathbf{T}_i)_{2 \leq i \leq q}$ as in (28).

According to Lemma 4.6,1. there exists for every $i \in \{1, \dots, q - 1\}$ and every $j \in \{0, \dots, f(i) - l_i - 1\}$ a subalgebra $\mathbf{S}_{i,j}$ and an isomorphism $f_{i,j} : \mathbf{S}_{i,j+1} \rightarrow \mathbf{T}_{i,j+1}$ over $S_{i,j}$, the sequences $(\mathbf{T}_{i,j})_{0 \leq j \leq f(i) - l_i - 1}$, $i = 1, \dots, q - 1$, as in (30).

According to Lemma 4.6,3. there exists for every $j \in \{0, \dots, p - 1\}$ a subalgebra \mathbf{S}_{2q+j+1} of \mathbf{P} and an isomorphism $f_{2q+j} : \mathbf{S}_{2q+j+1} \rightarrow \mathbf{U}_{j+1}$ over S_{2q+j} , the sequence $(\mathbf{U}_j)_{0 \leq j \leq p}$ as in (31).

The above implies that $f_{2q+p} : \mathbf{S}_{2q+p} \rightarrow \mathbf{T}$ is the desired isomorphism over S since $\mathbf{U}_p = \mathbf{T}$.

□

References

- [1] M. Albert, S. Burris, *Finite Axiomatizations for Existentially closed Posets and Semilattices*, Order. **3** (1986), 169–178
- [2] O. Frink, *Pseudo-complements in semilattices*, Duke Mathematical Journal **37**, (1962), 505–514
- [3] C. Gerber, J. Schmid, *The model companion of Stone semilattices*, Zeitschr. f. math. Logik und Grundlagen d. Math. **37**, (1991), 501–512
- [4] G. Grätzer, *General lattice theory*, 2nd ed., Birkhäuser, Basel 1998
- [5] G. Jones, *Pseudocomplemented semilattices*, PhD-thesis, unpublished
- [6] A. Macintyre, *Model completeness*, Handbook of Mathematical Logic (ed. J. Barwise), North-Holland, Amsterdam 1977, 139–180
- [7] R. Rupp, *On algebraically closed and existentially complete p-semilattices*, PhD Thesis, University of Bern, 2006
- [8] R. Rupp, J. Adler, J. Schmid, *The class of algebraically closed p-semilattices is finitely axiomatizable*, ??? ???, (???), ???
- [9] J. Schmid, *Algebraically closed distributive p-algebras*, Algebra Universalis **15**, (1982), 126–141
- [10] J. Schmid, *Model companions of distributive p-algebras*, The Journal of Symbolic Logic, Vol. **47** (1982), 680–688
- [11] J. Schmid, *Algebraically closed p-semilattices*, Arch. Math., Vol. **45** (1985), 501–510
- [12] P. Schmitt, *The model-completion of Stone algebras*, Ann. Sci. Univ. Clermont, Sr. Math., Vol. **13** (1976), 135–155